Coherent States: A General Approach

P. K. Panigrahi[1] [a,b], T. Shreecharan [a,b], J. Banerji [a] and V. Sundaram [c]

[a] Physical Research Laboratory, Navrangpura, Ahmedabad-380 009, India
[b] School of Physics, University of Hyderabad, Hyderabad-500 046, India
[c] Department of Electrical Engineering, IIT Madras, Chennai-600 036, India

Abstract

A general procedure for constructing coherent states, which are eigenstates of annihilation operators, related to quantum mechanical potential problems, is presented. These coherent states, by construction are not potential specific and rely on the properties of the orthogonal polynomials, for their derivation. The information about a given quantum mechanical potential enters into these states, through the orthogonal polynomials associated with it and also through its ground state wave function. The time evolution of some of these states exhibit fractional revivals, having relevance to the factorization problem.

1 Introduction

Coherent states (CS) were first introduced by Schrödinger [1], in the context of the harmonic oscillator, in an attempt to search for quantum states, whose time evolution followed classical equations of motion. These states were later studied exhaustively and expanded, because of their relevance to lasers and various other physical problems [2, 3]. The fact that the underlying algebraic structure of the harmonic oscillator and the single mode radiation field is the Heisenberg-Weyl algebra, naturally explains the relevance of the coherent state constructed by Schrödinger to lasers. The generalizations based on $su(1,1)$ and $su(2)$ algebras followed keeping the two mode radiation field in mind [4, 5].

CS are broadly separated into the following categories. The first type are the eigenstates of certain annihilation operators. The oscillator CS belong to this category, as also the so-called Barut-Girardello CS [6], which are the eigenstates of the annihilation operator of the $su(1,1)$ algebra. Based on the dynamical symmetry possessed by the physical system, a unitary operator $\hat{U}$ can be defined such that, its action on the ground state $\mid 0\rangle$ yields a class of states, known as the Perelomov CS [7]. States having minimum uncertainty product e.g., $\Delta x \Delta p = 1/2$ (in the unit $\hbar$=1) for the oscillator, are known as the minimum uncertainty CS. These type of CS have been generalized to a wide class of one dimensional exactly solvable potentials by Nieto and Simmons [8]. The maintainance of temporal stability under time evolution have recently led to a new class of CS [9]. As is well-known, for the harmonic oscillator all these definitions lead to the same coherent state.

[1]talk presented at the conference *Frontiers of Fundamental Physics*, held at B.M. Birla Science Center, Hyderabad, January 2003. To appear in the proceedings.

CS possess many interesting properties and have been extensively used to explain a wide range of physical phenomena, see [4] for an excellent review. In general, CS are not orthogonal to each other and provide an overcomplete basis set. Recently, there has been intense activity in the study of CS [10], largely inspired by the phenomena of wave packet revivals and fractional revivals in atomic systems [11]. The Rydberg atom wave packets provide one such example [12]. It is interesting to note that, some of these wave packets can be utilized for the factorization of numbers [13].

The phenomena of wave packet revival and fractional revival occurs in physical systems, whose energy has nonlinear terms, e.g., $E_n \propto n^2$. A number of quantum mechanical systems, e.g., square-well [14], Morse [15] and Pöschl-Teller [16] potentials, exhibit quadratic spectra. Wave packets related to potential wells have been studied exhaustivley [17], although the same is not true for the later two potentials. Hence, it is appropriate that one gains a better understanding of the CS of these potentials [10, 18, 19]. Here, we present a procedure to construct these CS in a general manner.

The paper is organized as follows: In the following section CS, for potentials with Laguerre polynomials as their eigenstates, modulo the measure, is constructed. In section 3, we derive the CS for those exactly solvable potentials, whose eigenstates can be expressed in terms of hypergeometric functions. These can be used to construct CS for Pöschl-Teller potential. We present our conclusions in section 4.

2 Coherent states for potentials of confluent hypergeometric class

In this section, we derive the coherent states for those exactly solvable potentials whose polynomial part of the eigenstates can be written in terms of the Laguerre polynomials. These potentials include Morse, Coulomb and three dimensional isotropic oscillator. For this purpose, we make use of a novel method of solving linear differential equations [20]. This method connects the space of solutions to the space of monomials.

The novel form of the solution for the Laguerre differential equation

$$\left[x\frac{d^2}{dx^2} + (\lambda + 1 - x)\frac{d}{dx} + n\right] L_n^\lambda(x) = 0 \tag{1}$$

is given by

$$L_n^\lambda(x) = \frac{(-1)^n}{n!} \exp\left[-x\frac{d^2}{dx^2} - (\lambda + 1)\frac{d}{dx}\right] . \, x^n \quad . \tag{2}$$

As pointed out earlier, this form of the solution relates the solution space to the space of monomials; this makes it easy to identify the ladder operators at the level of monomials. These operators can be cast into the ladder operators for Laguerre polynomials after a suitable similarity transformation. In the present work we do not provide the details of such construction, instead refer the readers to [21]. We identify

$$K_+ \equiv x \quad , \quad K_- \equiv \left[x\frac{d^2}{dx^2} + (\lambda+1)\frac{d}{dx}\right] \quad \text{and} \quad K_3 \equiv D + \frac{\lambda+1}{2} \quad , \tag{3}$$

as the raising, lowering and the diagonal operator respectively at the level of monomials:

$$K_+x^n = x^{n+1}\,,\; K_-x^n = n(n+\lambda)x^{n-1}\,,\; K_3x^n = \left[n + \frac{\lambda+1}{2}\right]x^n \quad , \tag{4}$$

which satisfy

$$[K_+, K_-] = -2K_3 \qquad [K_3, K_\pm] = \pm K_\pm \quad , \tag{5}$$

a $su(1,1)$ algebra. Having identified the necessary ladder operators and the algebra satisfied by them, we now proceed to construct the coherent state. Here, we construct the eigenstate of the lowering operator K_-. For this purpose, we first identify, the operator $\tilde{K}_+$ such that $[K_-, \tilde{K}_+] = 1$, utilizing the procedure developed by Shanta et. al. [22]. The explicit expression of $\tilde{K}_+$ in the present case is

$$\tilde{K}_+ = \frac{1}{(D+\lambda)}x \quad . \tag{6}$$

Defining $U = \exp(\alpha\tilde{K}_+)$, and introducing an identity operator and acting U^{-1} from left in $K_- \mid 0\rangle = 0$, one gets

$$U^{-1}K_-UU^{-1} \mid 0\rangle = 0 \quad . \tag{7}$$

Since $U^{-1}K_-U = (K_- + \alpha)$ and $K_-\langle x \mid 0\rangle = 0$ yields $\langle x \mid 0\rangle = x^0$, the coherent state $\mid \alpha\rangle$ in co-ordinate basis is given by

$$\langle x \mid \alpha\rangle = N(\alpha)^{-1}S^{-1}e^{-\alpha\tilde{K}_+}x^0 \quad . \tag{8}$$

Here we have introduced $S^{-1} = \exp(-K_-)$ for the sake of future convenience; this affects only the normalization. One then gets

$$\langle x \mid \alpha\rangle = N(\alpha)^{-1}S^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\alpha\tilde{K}_+)^n x^0 \quad , \tag{9}$$

here $N(\alpha)^{-1}$ is the normalization factor. Writing explicitly

$$\langle x \mid \alpha\rangle = N(\alpha)^{-1}S^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(\alpha x)^n}{(D+\lambda+n)\cdots(D+\lambda+1)}\,.\,x^0\,. \tag{10}$$

or

$$\langle x \mid \alpha\rangle = N(\alpha)^{-1}S^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{(\alpha x)^n}{(\lambda+1)_n} \quad . \tag{11}$$

Here, $(\lambda+1)_n$ is the Pocchammer symbol and can be defined in terms of the gamma functions

$$(\lambda+1)_n = \frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda+1)} \quad . \tag{12}$$

Substituting Eq. (12) and Eq. (2) in Eq. (11), we obtain

$$\langle x \mid \alpha \rangle = N(\alpha)^{-1}\Gamma(\lambda+1)\sum_{n=0}^{\infty}\frac{\alpha^n L_n^\lambda(x)}{\Gamma(\lambda+n+1)} \quad . \tag{13}$$

It can be easily noticed that the coherent state constructed above does not contain, *a priori*, any information about the potential. This can be incorporated by muliplying the groundstate wave function, specific to the potential under consideration, from the left. The above equation can be cast into a compact form utilizing the generating function of the Laguerre polynomials [23],

$$\langle x \mid \alpha \rangle = N(\alpha)^{-1}\Gamma(\lambda+1)(x\alpha)^{-\lambda/2}e^{\alpha}J_\lambda[2(x\alpha)^{1/2}] \quad . \tag{14}$$

here $J_\lambda[2(x\alpha)^{1/2}]$ is the Bessel function of the first kind. Although the procedure outlined here is new, this coherent state has been derived earlier [24, 25].

3 Coherent state of modified Pöschl-Teller potential

In this section, we derive the coherent state for the modified Pöschl-Teller potential following the technique developed in the previous section. For this purpose, we first start with the hypergeometric differential equation and write its solution in the form which can be used easily for constructing the coherent state. The hypergeometric differential equation is

$$\left[z^2\frac{d^2}{dz^2}+(a+b+1)z\frac{d}{dz}+ab-z\frac{d^2}{dz^2}-c\frac{d}{dz}\right]F(a,b;c;z)=0 \quad , \tag{15}$$

whose solution can be written in the novel form

$$F(a,b;c;z) = (-1)^{-a}\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)}\exp\left[\frac{-1}{(D+b)}\left(z\frac{d^2}{dz^2}+c\frac{d}{dz}\right)\right] . \, z^{-a} \quad . \tag{16}$$

The above form makes easy the identification of the underlying algebraic structure. Defining the operators

$$K_- = \frac{1}{(D+b)}\left(z\frac{d^2}{dz^2}+c\frac{d}{dz}\right) \quad \text{and} \quad \tilde{K}_+ = \frac{(D+b-1)}{(D+c-1)}z \tag{17}$$

such that they satisfy the algebra $[K_-, \tilde{K}_+] = 1$. We obtain $\tilde{U} \equiv \exp(q\tilde{K}_+)$, which will be useful for finding the eigenstate of K_- operator. Starting from $K_- \mid 0\rangle = 0$, and proceeding, as shown earlier for the case of Laguerre polynomials, we get

$$\tilde{U}^{-1}K_-\tilde{U}\tilde{U}^{-1} \mid q\rangle = 0 \quad , \quad (K_- + q)\tilde{U}^{-1} \mid q\rangle = 0 \quad . \tag{18}$$

For casting the above into a convenient form, we define, $\tilde{S}^{-1} \equiv \exp(-K_-)$ which yields,

$$(K_- + q)\tilde{S}^{-1}\tilde{U}^{-1} \mid q\rangle = 0 \quad . \tag{19}$$

The above coherent state $\tilde{S}^{-1}\tilde{U}^{-1} \mid q\rangle$ in the cordinate basis is given by

$$\begin{aligned} \langle z \mid q\rangle &= e^{-K_-}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}q^n\left[\frac{D+b-1}{D+c-1}z\right]^n z^0 \quad \text{or} \\ \langle z \mid q\rangle &= e^{-K_-}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}q^n z^n\frac{(D+b+n-1)\cdots(D+b)}{(D+c+n-1)\cdots(D+c)}z^0\,, \end{aligned} \tag{20}$$

and hence,

$$\langle z \mid q\rangle = \sum_{n=0}^{\infty}\frac{(-q)^n}{n!}\frac{\Gamma(b+n)\Gamma(c)}{\Gamma(c+n)\Gamma(b)}e^{-K_-}z^n \quad . \tag{21}$$

Utilizing the expression for $F(-n, b, c; z)$, the coherent state can be expressed as

$$\langle z \mid q\rangle = N(q)^{-1}\sum_{n=0}^{\infty}\frac{q^n}{n!}F(-n, b; c; z) \quad , \tag{22}$$

where $N(q)^{-1}$ is the normalization constant. Eigenfunctions of the modified Pöschl-Teller potential [26] are Gegenbauer polynomials, which is a special case of hypergeometric series with specific parameter values: $b = n + 2\rho$ and $c = \rho + 1/2$. With the above substitutions the coherent state is given by

$$\langle z \mid q\rangle = N(q)^{-1}\sum_{n=0}^{\infty}\frac{\Gamma(2\rho)}{\Gamma(2\rho+n)}q^n C_n^{\rho}(1-2z) \quad , \tag{23}$$

where we have used the relation

$$F(-n, n+2\rho; \rho+1/2; z) = \frac{n!}{(2\rho)_n}C_n^{\rho}(1-2z) \quad . \tag{24}$$

The coherent state derived above can be written in a compact form using the generating function for the Gegenbauer polynomials [23]:

$$\langle y \mid q\rangle = N(q)^{-1}\Gamma(\rho+\frac{1}{2})\exp(q\cos\theta)\left[\frac{q}{2}\sin\theta\right]^{\frac{1}{2}-\rho}J_{\frac{1}{2}-\rho}(q\sin\theta) \quad , \quad . \tag{25}$$

where $(1-2z) = y = \cos\theta$. The fact that modified Pöschl-Teller potential has a quadratic spectra, leads to revivals and fractional reivals during the time evolution of the above CS. This is transparent in the auto-correlation function plotted below. It is interesting to note that the above phenomena has recently been used for the factorization of numbers.